Journal of Engineering Mathematics, Vol. 6, No. 2, April 1972
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

The Indefinite Z-Transform Technique and Application to Analysis of Difference Equations

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(Received June 1, 1970 and in revised form October 1, 1971)

SUMMARY

The indefinite Z-transform technique is proposed. The method for solving linear difference equations using indefinite Z-transforms is compared with the methods employing the infinite one-sided Z-transforms and the finite Z-transforms. The distinct advantage of the method presented in this paper is that the desired solutions are obtained without employing standard inverse Z-transform techniques, such as the convolution theorem, or extensive Z-transform tables. All that is needed here for the derivations of the desired solutions is the application of Cramer's rule for the solution of simultaneous algebraic equations using the characteristic values. Therefore, this technique could be also readily used by those who have not studied the familiar Z-transform technique.

1. Introduction

Problems that are characterized by ordinary linear difference equations with constant coefficients and known initial conditions are commonly solved using the familiar, infinite Z-transforms [1]. This approach is particularly suitable because the initial conditions are incorporated into the transformed set of difference equations and the desired solutions are derived directly upon inversion of the Z-transforms. The infinite Z-transform techniques can also be used to derive the solutions of boundary-value problems characterized by linear difference equations such as discrete electrostatic field problems and ladder type networks. In these cases, however, it is necessary to assume a set of initial boundary conditions and only after determining the inverse transforms, are the unknown initial conditions evaluated by imposing the as yet unused final boundary conditions. These values for the initial conditions are finally substituted into the expressions for the inverse transforms.

The general theory of finite Z-transforms advanced by Higgins and Oesterlei [2], [3] has more recently been employed to derive the solution to the boundary value problems characterized by linear difference equations. Using these finite transforms it is demonstrated that it is possible to incorporate both the initial and final boundary conditions into the transformed difference equations. In this case, before the inverse transforms are evaluated, it is necessary to determine a complementary set of boundary conditions that are introduced in the course of transforming the given difference equations. This complementary set of boundary conditions are determined by imposing the conditions for the analyticity of finite Z-transforms. Finally, through the use of a new set of tables for finite Z-transforms, (or by inspection of the terms of the finite Z-transform and the use of familiar infinite Z-transform tables), the desired solutions may be derived.

It is to be noted that both the techniques considered above require (a), the determination of unspecified boundary conditions and (b) the determination of the respective inverse transforms. Thus, it is not possible to yield an overall judgment as to the preference of one technique over the other. For instance while the finite transform technique is somewhat more elegant, it may well be that it also requires the solution of more undetermined boundary conditions, than would be required if the infinite Z-transform techniques were used. However, if one is only interested in determining the values of the dependent variables at the boundaries, the finite transform technique may be preferred, since no inverse transforms are sought.

In this paper, indefinite Z-transform techniques are proposed. They are employed to derive solutions of difference equations with either final or initial boundary conditions. For the case in which all the initial boundary conditions are known, the solution is determined directly by imposing the analyticity conditions for the indefinite Z-transforms, without actually seeking out the inverse transform through the use of tables or by other familiar techniques.

For the case in which some of the initial boundary conditions are unknown, and in their place certain final boundary conditions are specified, one may proceed in one of the two following ways. As in the case in which the infinite Z-transforms are employed, initial conditions are assumed. The solutions in terms of these initial boundary conditions are obtained by requiring that the indefinite Z-transforms be analytical. The unknown initial boundary conditions are finally determined by employing all the remaining unused final boundary conditions. Alternatively, one may choose to proceed with the solution (as in the cases when the finite Z-transforms are used) by first determining the unknown initial boundary conditions. After these are determined, the desired solutions are obtained by imposing the analyticity conditions for the indefinite Z-transform.

Hence, while the indefinite Z-transform technique presented here retains the flexibility discussed above in the manner one chooses to determine the unspecified boundary conditions, it yields an additional advantage over the earlier methods in that it is not necessary to employ standard inversion techniques or extensive infinite Z-transform tables.

In view of the well-known parallels between the Z-transform techniques for difference equations and the Laplace transform techniques for differential equations, it can readily be shown that indefinite Laplace transform techniques similar to the indefinite Z-transform techniques proposed here, may be devised to solve initial and final boundary value problems characterized by linear differential equations.

A historical survey of various operational calculus and transform techniques used in the past to solve problems in electric circuit theory is given by Higgins [1]. The key to the finite and indefinite Laplace and Z-transform techniques is the recognition of the necessary condition of the analyticity of these transforms. Various sets of necessary and sufficient conditions for the existence of the finite Laplace transform are presented by Doetsch in his treatise on the Laplace transform [4].

2. Definition of the Finite and Indefinite Z-Transforms and Their Relationship to the One Sided Infinite Z-Transforms

Let $f(n)$ be defined over the range $n=0, 1, 2, \ldots$ The familiar, one sided infinite Z-transform of $f(n)$ is defined as follows.

$$Z[f(n)] = F(z) = \sum_{p=0}^{\infty} f(p) z^{-p} . \tag{2.1}$$

Let $f(n)$ be given over the range $n=0, 1, \ldots, N$. The finite one sided Z-transform of $f(n)$ is defined by the following finite sum,

$$Z_D[f(n)] = F_D(z) = \sum_{p=0}^{N} f(p) z^{-p} \qquad p = 0, 1, 2, \ldots, N . \tag{2.2}$$

Like $F(z)$, the definite Z-transform $F_D(s)$ is only a function of the variable z. However, the indefinite Z-transform is a function of both z and the variable integer n. It is defined as follows.

$$Z_I[f(n)] = F_I(n, z) = \sum_{p=0}^{n} f(p) z^{-p} \qquad n = 0, 1, 2, \ldots, N . \tag{2.3}$$

Hence,

$$f(n) = z^n [F_I(n, z) - F_I(n-1, z)] .$$

The range of n may be chosen to be finite or infinite depending upon the range of the integer n in the problem considered. It is now obvious from the above definitions that

$$F_I(N, z) = F_D(z)\,. \tag{2.4}$$

Furthermore, it can readily be shown that

$$Z_D[f(n)] = Z[f(n)] - \sum_{N+1}^{\infty} f(p)z^{-p} = Z[f(n)] - z^{-(N+1)} Z[f(n+N+1)]\,. \tag{2.5}$$

As a simple application of the above relationships, one can determine $F_D(z)$ and $F_I(n, z)$ for the function $f(n) = \exp\{(i\omega - \alpha)n\}$. The infinite Z-transform for $\exp\{(i\omega - \alpha)n\}$ is

$$Z[\exp\{(i\omega-\alpha)n\}] = z/(z - e^{i\omega-\alpha})\,. \tag{2.6}$$

Thus using (2.5) and (2.6),

$$\begin{aligned} Z_D[\exp\{(i\omega-\alpha)n\}] &= Z[\exp\{(i\omega-\alpha)n\}][1 - z^{-(N+1)} e^{(i\omega-\alpha)(N+1)}] \\ &= (z - z^{-N} e^{(i\omega-\alpha)(N+1)})/(z - e^{i\omega-\alpha})\,. \end{aligned} \tag{2.7}$$

It is to be noted that while $Z[\exp\{(i\omega-\alpha)n\}]$ is singular for $z = e^{i\omega-\alpha}$, the expression for $Z_D[\exp\{(i\omega-\alpha)n\}]$ is analytical in the complex z plane except at the origin. The indefinite transform may now be obtained by simply substituting the variable integer n for the integer N in (2.5) as indicated by (2.4). A large set of finite and indefinite Z-transforms may be derived from (2.7) by noting the relationship between the sinusoidal and hyperbolic functions and the exponential functions. Also by setting $i\omega - \alpha = 0$ in (2.7) one obtains directly the Z-transform for the unit step function $u(n)$. Another special result that may be deduced from (2.5) is the definite (or indefinite) transform of $f(n+1)$. Thus noting that $Z[f(n+1)] = zF(z) - zf(0)$, it follows that

$$Z_D[f(n+1)] = Z[f(n+1)] - z^{-(N+1)} Z[f(n+1+N+1)] = zF_D(z) - zf(0) + z^{-N} f(N+1)\,. \tag{2.8a}$$

In the same way it can be shown that

$$Z_D[f(n+2)] = z^2 F_D(z) - z^2 f(0) - zf(1) + z^{-N+1} f(N+1) + z^{-N} f(N+2)\,. \tag{2.8b}$$

Similarly, to obtain the respective indefinite Z-transforms replace N by n in the above expressions.

As a final note in this section, it is easily verified that provided $f(n)$ are bounded functions of n, such that $|f(p)| < B$, $p = 0, 1, \ldots n$ indefinite Z-transforms are bounded functions of n and z except for $z = 0$, since

$$\left|\sum_{p=0}^{n} f(p) z^{-p}\right| \leq B \sum_{p=0}^{n} |z|^{-p} = B(|z| - |z|^{-n})/(|z| - 1)\,.$$

3. Demonstration and Comparison of the Z-transform Techniques

To demonstrate and compare the three transform techniques discussed in this paper, consider the following problem characterized by a simple difference equation.

$$\Delta a(n) + ca(n) = a(n+1) - a(n) + ca(n) = 0\,, \quad a(M) = a_M\,, \quad n = 0, 1, 2, \ldots\,. \tag{3.1}$$

in which the range of n is infinite, M is a constant integer, and a_M is a given constant. Consider, at first, the infinite Z-transform technique. Let $Z[a(n)] = A(z)$ and assume the unknown initial condition $a(0) = a_0$. The Z-transform of (3.1) yields the following equation.

$$A(z) = a_0 z/(z - 1 + c)\,. \tag{3.2}$$

The inverse transform is found in tables for Z-transforms. Thus,

$$a(n) = Z^{-1}[A(z)] = a_0(1-c)^n\,. \tag{3.3}$$

Imposing the unused boundary condition on (3.3) yields

$$a_0 = (1-c)^{-M} a_M\,. \tag{3.4}$$

Thus finally the solution is obtained by substituting (3.4) into (3.3)

$$a(n) = a_M (1-c)^{n-M}\,. \tag{3.5}$$

Now consider the definite transform method. Here define $Z_D[a(n)] = A_D(z)$ and as above $a(0) = a_0$. The definite transform of (3.1) (taking $N+1 = M$ in 2.8a) yields the following equation.

$$A_D(z) = (a_0 z - a_M z^{-M+1})/(z-1+c)\,. \tag{3.6}$$

Noting that $A_D(z)$ is finite for $z = 1-c$, it is necessary that the nominator in (3.6) vanishes for $z = 1-c$. Thus

$$a_0 = a_M (1-c)^{-M}\,. \tag{3.7}$$

The solution may now be derived by inverting (3.6) and using (3.7). Thus, using a table for definite Z-transforms or using tables for the infinite transforms in conjunction with (2.5), the desired solution is obtained

$$Z_D^{-1}[A_D(z)] = Z_D^{-1}[a_M z\{(1-c)^{-M} - z^{-M}\}/(z-1+c)] = a_M (1-c)^{n-M}\,. \tag{3.8}$$

Next, consider the indefinite transform method. Define $Z_I[a(n)] = A_I(n, z)$ and $a(0) = a_0$ as before. The indefinite transform of (3.1) produces the following equation.

$$(z-1+c)\,A_I(n, z) = (a_0 z - a(n+1) z^{-n})\,. \tag{3.9}$$

By imposing the analyticity condition for $A_I(n, 1-c)$ and substituting n for $n+1$, one obtains from (3.9)

$$a(n) = a_0 (1-c)^n\,. \tag{3.10}$$

Now, set $n = M$ in (3.9) to determine a_0 in terms of a_M (3.4). Note that the desired solution is determined in this case without directly seeking the inverse transform of $A_I(n, z)$ in tables.

In the following section, the indefinite transform technique is illustrated by solving the current in all the loops of a ladder type network. Such problems are solved in several textbooks using the infinite Z-transform techniques [5]. The mode amplitudes in a lumped transmission line system with uniform mode coupling and initial conditions are derived in Section 5.

4. Solution of Linear Difference Equations with Mixed Boundary Conditions and Arbitrary Driving Functions

Consider the $N+1$ loop ladder network shown in figure 1. The net enforced EMF in the nth loop, $e(n)$, is an arbitrary function of n (not shown in Fig. 1). The series and parallel impedances are

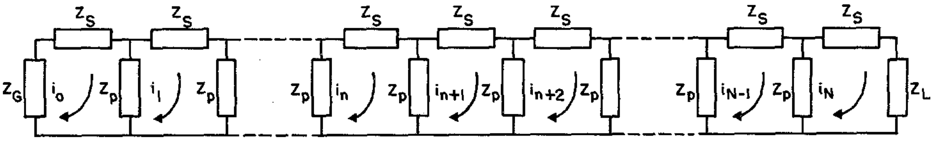

Figure 1.

Z_s and Z_p respectively. The $N+1$ difference equations characterizing the network are

$$i(1) = \beta i(0) - Y_p e(0)\,; \quad \beta = 1 + Y_p(Z_s + Z_G)\,, \quad Y_p = 1/Z_p\,, \tag{4.1}$$

$$i(n) - \delta i(n+1) + i(n+2) = -Y_p e(n+1)\,; \quad \delta = 2 + Y_p Z_s \text{ for } n = 0, 1, 2, \ldots N-2\,, \tag{4.2}$$

and

$$i(N-1) = \gamma i(N) - Y_p e(N)\,; \quad \gamma = 1 + Y_p(Z_s + Z_L) \tag{4.3}$$

in which Z_G and Z_L are the impedances in loops 0 and N respectively. Take the indefinite

Z-transform of (4.2) and let I and E be the symbols used here for the indefinite transforms for $i(n)$ and $e(n)$ respectively. On collecting like terms one gets, for $n=0, 1, \ldots, N-2$,

$$(1-\delta z+z^2)I$$
$$=z(z-\delta)i(0)+zi(1)-z^{-n}(z-\delta)i(n+1)-z^{-n}i(n+2)-Y_p[zE-ze(0)+z^{-n}e(n+1)]\,. \quad (4.4)$$

The roots of the characteristic equation, $z^2-\delta z+1=0$, are

$$z_1=\mathrm{e}^{\alpha} \text{ and } z_2=\mathrm{e}^{-\alpha}\,; \quad 2\cosh\alpha\equiv\delta\,. \quad (4.5)$$

Since I is finite for $z=z_1$ and $z=z_2$, the right-hand side of (4.4) must vanish for $z=z_1$ and $z=z_2$. Thus

$$i(n+1)-i(n+2)\,\mathrm{e}^{\alpha}$$
$$=i(0)\,\mathrm{e}^{(n+1)\alpha}-i(1)\,\mathrm{e}^{(n+2)\alpha}+Y_p[E(n,\mathrm{e}^{\alpha})\,\mathrm{e}^{(n+2)\alpha}-e(0)\,\mathrm{e}^{(n+2)\alpha}+e(n+1)\,\mathrm{e}^{\alpha}] \quad (4.6a)$$

and

$$i(n+1)-i(n+2)\,\mathrm{e}^{-\alpha}$$
$$=i(0)\mathrm{e}^{-(n+1)\alpha}-i(1)\mathrm{e}^{-(n+2)\alpha}+Y_p[E(n,\mathrm{e}^{-\alpha})\mathrm{e}^{-(n+2)\alpha}-e(0)\mathrm{e}^{-(n+2)\alpha}+e(n+1)\mathrm{e}^{-\alpha}]\,. \quad (4.6b)$$

Subtract (4.6a) from (4.6b), use (4.1) to eliminate $i(1)$ and substitute n for $n+2$ in the resulting equation to get the desired solution for $i(n)$ in terms of $i(0)$.

$$i(n)=\frac{1}{\sinh\alpha}\left\{i(0)[\beta\sinh n\alpha-\sinh(n-1)\alpha]-\frac{Y_p}{2}[E(n,\mathrm{e}^{\alpha})\mathrm{e}^{n\alpha}-E(n,\mathrm{e}^{-\alpha})\mathrm{e}^{-n\alpha}]\right\}. \quad (4.7a)$$

Finally, use (4.3) in conjunction with (4.7a) to solve for $i(0)$.

$$i(0)=\frac{[E(N,\mathrm{e}^{\alpha})\,\mathrm{e}^{N\alpha}(\gamma-\mathrm{e}^{-\alpha})-E(N,\mathrm{e}^{-\alpha})\,\mathrm{e}^{-N\alpha}(\gamma-\mathrm{e}^{\alpha})]\,Y_p/2}{\sinh(N-2)\alpha-(\beta+\gamma)\sinh(N-1)\alpha+\beta\gamma\sinh N\alpha}\,. \quad (4.7b)$$

To derive the solution using the regular Z-transform techniques, set $n\to\infty$ in (4.4) and invert the resulting equation. To do this, use Z-transform tables in conjunction with the convolution theorem for the inversion of a product of Z-transforms. To eliminate $i(0)$ and $i(1)$ from the result, use (4.1) and (4.3) as in the above solution. Note however, in the solution (4.7), the convolution sum is given in closed form. To illuminate this final point further, set

$$e(n)=u(n)\,. \quad (4.8)$$

Thus, using (2.7) (with $\mathrm{i}\omega-\alpha=0$),

$$\mathrm{e}^{n\alpha}E(n,\mathrm{e}^{\alpha})=\frac{\mathrm{e}^{(n+1)\alpha}-1}{\mathrm{e}^{\alpha}-1}=\mathrm{e}^{n\alpha/2}\cdot\frac{\sinh((n+1)\alpha/2)}{\sinh(\alpha/2)}\,. \quad (4.9)$$

The last term in (4.7a) and the nominator in (4.7b) reduce to

$$Y_p\sinh((n+1)\alpha/2)\sinh(n\alpha/2)/\sinh(\alpha/2)$$

and

$$Y_p[\gamma\sinh(N\alpha/2)-\sinh((N-2)\alpha/2)]\sinh((N+1)\alpha/2)/\sinh(\alpha/2)\,.$$

Using infinite Z-transform techniques to solve the same problem, the following expressions are obtained instead by employing the convolution theorem

$$Y_p\sum_0^n e(p)\sinh(n-p)$$

and

$$Y_p\sum_0^N e(p)\,[\gamma\sinh(N-p)\alpha-\sinh(N-1-p)\alpha]\,.$$

These expressions can be shown to be identical to the proceeding closed form expressions derived directly using (4.7). Since it is not necessary to resort to standard inverse Z-transform techniques to obtain the desired solution, the indefinite Z-transform method is particularly useful for solving problems in which the inverse transforms are not commonly found in Z-transform tables.

Finally, it should be noted that if for example, in the network considered in this section, one is only interested in determining the currents in the boundary loops, 0 and N, the derivation of the solutions using finite Z-transform techniques is preferable to the derivation involving the infinite Z-transforms, since in the former case, no inverse transforms are sought. In this case the derivations using the finite or indefinite transform techniques are indistinguishable.

5. Solution of Coupled Homogeneous Difference Equations

The wave amplitudes $a_1(n)$ and $a_2(n)$ in a coupled transmission line with lumped parameters are characterized by the following difference equations and boundary conditions respectively,

$$\left.\begin{aligned} &\Delta a_1(n)+k_{11}a_1(n)+k_{12}a_2(n)=0\\ &\Delta a_2(n)+k_{21}a_1(n)+k_{22}a_2(n)=0\,,\\ \text{and}\quad &\\ &a_1(0)=a_{10}\,,\ a_2(0)=a_{20}\,,\end{aligned}\right\} \tag{5.1}$$

in which k_{11} and k_{22} are constant propagation coefficients. The uniform coupling coefficients are k_{12} and k_{21}.

To solve for $a_1(n)$ and $a_2(n)$, we first write the indefinite Z-transforms for the above difference equation in matrix notation.

$$\begin{bmatrix} (z-1+k_{11}) & k_{12}\\ k_{21} & (z-1+k_{22})\end{bmatrix}\begin{bmatrix} A_1\\ A_2\end{bmatrix} = z\begin{bmatrix} a_{10}-a_1(n)z^{-n}\\ a_{20}-a_2(n)z^{-n}\end{bmatrix} \tag{5.2}$$

Solution of the above equation for the indefinite Z-transforms A_1 and A_2 yields,

$$[(z-1+k_{11})(z-1+k_{22})-k_{12}k_{21}]\begin{bmatrix} A_1\\ A_2\end{bmatrix} = z\begin{bmatrix} z-1+k_{22} & -k_{12}\\ -k_{21} & z-1+k_{11}\end{bmatrix}\begin{bmatrix} a_{10}-a_1(n)z^{-n}\\ a_{20}-a_2(n)z^{-n}\end{bmatrix} \tag{5.3}$$

Since the indefinite Z-transforms A_1 and A_2 are analytic functions of z, the matrix product on the right-hand side of (5.3) must equal the zero column vector for $z=z_1$ and $z=z_2$; the roots of the characteristic equation for this problem.

$$z_1=1-(\alpha+\beta)\,,\quad z_2=1-(\alpha-\beta)\,,$$

where

$$\alpha=\frac{k_{11}+k_{22}}{2}\,,\quad \beta=\left[\left(\frac{k_{11}-k_{22}}{2}\right)^2+k_{12}k_{21}\right]^{\frac{1}{2}}. \tag{5.4}$$

Thus on imposing the analyticity conditions for $A_1(n,z)$ one obtains the following equations.

$$\begin{aligned}&(z_1-1+k_{22})[a_{10}z_1^n-a_1(n)]=k_{12}[a_{20}z_1^n-a_2(n)]\\ &(z_2-1+k_{22})[a_{20}z_2^n-a_2(n)]=k_{12}[a_{20}z_2^n-a_2(n)]\,.\end{aligned} \tag{5.5}$$

The above algebraic equations may now be solved for $a_1(n)$ and $a_2(n)$ without employing the standard inverse Z-transform techniques. Thus,

$$\left.\begin{aligned} a_1(n) &= \frac{a_{10}}{2\beta}\{\beta[z_2^n+z_1^n]+(k_{22}-\alpha)[z_2^n-z_1^n]\}-\frac{a_{20}}{2\beta}k_{12}[z_2^n-z_1^n]\\ \text{and}\quad &\\ a_2(n) &= \frac{a_{20}}{2\beta}\{\beta[z_2^n+z_1^n]+(k_{11}-\alpha)[z_2^n-z_1^n]\}-\frac{a_{10}}{2\beta}k_{21}[z_2^n-z_1^n]\,.\end{aligned}\right\} \tag{5.6}$$

For the degenerate case, $z_1 = z_2 = 1-\alpha$, $(\beta = 0)$, the solutions may be derived by taking the limits of (5.6) as $\beta \to 0$. However, in order to demonstrate how the analyticity conditions are imposed for the cases in which the characteristic equation yields multiple roots, we reconsider the expressions for the indefinite transforms as $\beta \to 0$. Thus for A_1 we now have,

$$(z-1+\alpha)^2 A_1 = z\{z-1+k_{22})(a_{10}-a_1(n)z^{-n})-k_{12}(a_{20}-a_2(n)z^{-n})\} \equiv zF(z,n)\,. \quad (5.7)$$

The analyticity conditions for A_1 are now established by equating to zero $F(z_1, n)$ and $F'(z_1, n)$ (the first two terms of the Taylor expansion of $F(z, n)$ defined in (5.7)). Thus,

$$\left.\begin{aligned} &(z_1-1+k_{22})(a_{10}-a_1(n)z_1^{-n})-k_{12}(a_{20}-a_2(n)z^{-n}) = 0\,, \\ \text{and} \quad & \\ &a_{10}+[n(z_1-1+k_{22})-z_1]a_1(n)z_1^{-(n+1)}-nk_{12}a_2(n)z_1^{-n(n+1)} = 0\,. \end{aligned}\right\} \quad (5.8)$$

These linear algebraic equations can now be solved for $a_1(n)$ and $a_2(n)$.

$$a_1(n) = z_1^n \left\{ a_{10}\left[1+\frac{n}{2}(k_{22}-k_{11})z_1^{-1}\right] - a_{20}nk_{12}z_1^{-1} \right\}$$

and

$$a_2(n) = z_1^n \left\{ a_{20}\left[1+\frac{n}{2}(k_{11}-k_{22})z_1^{-1}\right] - a_{10}nk_{21}z_1^{-1} \right\}. \quad (5.9)$$

Similarly, for the case in which the multiplicity of the roots of the characteristic equation is r, to establish the analyticity of the transforms it is necessary to equate to zero the first r terms of the appropriate Taylor series expansion. In general, since the order of the set of difference equations characterizing the problem is the order of the characteristic equation, there are sufficient analyticity conditions to determine all the unknown functions.

6. Concluding Remarks

Using the finite Z-transform techniques, the formulation of the solution of linear difference equations with final boundary conditions is more elegant than the formulation using the standard infinite Z-transform techniques since all the boundary conditions are incorporated into the transformed equations. However, this is often at the expense of an actual increase in intermediate computations. There is, however, a clear advantage in employing indefinite Z-transforms, since the solutions are obtained without employing standard techniques for the inversion of the transforms such as the use of contour integrals, convolution sums or extensive Z-transform tables. This advantage pertains to problems with specified initial conditions (where the infinite Z-transforms are employed) as well as to problems with final boundary conditions. To use the indefinite Z-transform method it is only necessary to apply Cramer's rule for the solution of simultaneous algebraic equations using the characteristic values.

For problems in which only the values of the dependent variables at the boundaries are sought, finite Z-transform techniques are preferable to infinite Z-transform techniques, since no inverse transforms are required using finite transform techniques. In this case, the derivations of the solution using the finite or indefinite Z-transform techniques are indistinguishable.

The well-known parallels between the Z-transform techniques for difference equations and the Laplace transform techniques for differential equations, indicate that indefinite Laplace transform techniques similiar to the indefinite Z-transform techniques presented here, may be employed to solve initial and final boundary value problems characterized by linear differential equations [6].

Finally, Higgins and Oesterlei [2] have demonstrated the use of multiple finite transforms in solving problems with two independent variables. It is suggested here that multiple indefinite transform techniques may also be employed to solve these problems.

7. Acknowledgements

The author wishes to thank W. L. Brogan and F. G. Ullman for their comments.

REFERENCES

[1] T. J. Higgins, *History of the operational calculus and transform analysis as used in electric circuit theory*, Part I, Electrical Engineering, 68 (1949) 42–48, Part II 1948–1960, Paper No. C.P. 60-1928, AIEE Fall General Meeting, Chicago Ill., Oct. 10–14, 1960.

[2] T. J. Higgins and R. E. Oesterlei, *General theory of finite Z-transforms and application to analysis of discrete circuits*, IEEE summer general meeting and Nuclear Radiation Effects Conference, Toronto, Ont. Canada, June 16–21, 1963.

[3] R. E. Oesterlei, *Solutions of initial value and boundary value problems in electric circuits and electromagnetic field theory by finite Laplace and Z-transforms*, M.S. Thesis, Dept. of Elec. Engr., Univ. of Wisconsin, Madison, Wisconsin.

[4] G. Doetsch, *Handbuch der Laplace-Transformation*, Birkhauser Verlag, Basel, Switzerland, 3 Vols., 1956.

[5] J. A. Aseltine, *Transform method in linear system analysis*, McGraw-Hill, New York, 1958.

[6] E. Bahar, The indefinite Laplace transform technique and application to analysis of initial and boundary value problems, to be published in the International Journal of Electrical Engineering Education, Vol. 9, 1971.

Journal of Engineering Math., Vol. 6 (1972) 125–132